The Effects of Risk Aversion and Age on Investments in New Firms

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Abstract

How does the age of risk-averse individuals affect investments in private projects? This question is analyzed under the assumption that such individuals have to invest a large fraction of their personal wealth in order to establish a new firm. Thresholds conditional on age that trigger a switch from one asset to another are derived in a continuous-time portfolio choice model with zero-or-one choices. As the investor gets older – and depending on random events – not only wealth, but also the thresholds, might increase. Thus, there are two counteracting effects on the likelihood of becoming an entrepreneur.

Keywords: entrepreneurs, portfolio choice, either-or, horizon, risk aversion.

JEL: G11, D81, J2.

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1 Introduction

The effect of a credit constraint has been examined in several empirical studies on the determinants of the likelihood of becoming self-employed.[1] In most instances, panel data on the micro level were used to estimate the probability of entering into self-employment, conditional on explanatory variables measuring "wealth" (e.g., asset holdings, income variables, inheritance) and human capital (e.g., education, work experience, age). These studies found that personal assets have a positive effect on the likelihood of becoming self-employed, and concluded that this is brought about by binding credit constraints.[2] The effect of individuals' age on entry varied across these studies.

The distribution of entrepreneurs across age groups in Sweden for three different years is shown in Figure 1. Of course, while it does not reveal the flows, i.e., entry and exit, the striking aspect is that the distribution is similar over time even though the years represent very different states in the business cycle. For example, unemployment averaged 2.1 percent in 1987, 5.3 percent in 1992 and 8.0 percent in 1997.[3] Hence, there are regularities on the aggregate level but large gross flows underlying these net figures (cf. Persson (1999)).

Risk taking is associated with the establishment of small firms. King and Leape (1998) found that households hold only a limited number of different financial assets and that age strongly increases the likelihood of holding risky financial assets. Moreover, households with low wealth tend to hold a large proportion of riskfree assets. Age and wealth are positively correlated, however. These considerations imply that an analysis based on a dynamic theoretical micro model could provide a suitable means of proposing causalities, so as to draw conclusions about the relationship across age, wealth and risk taking.

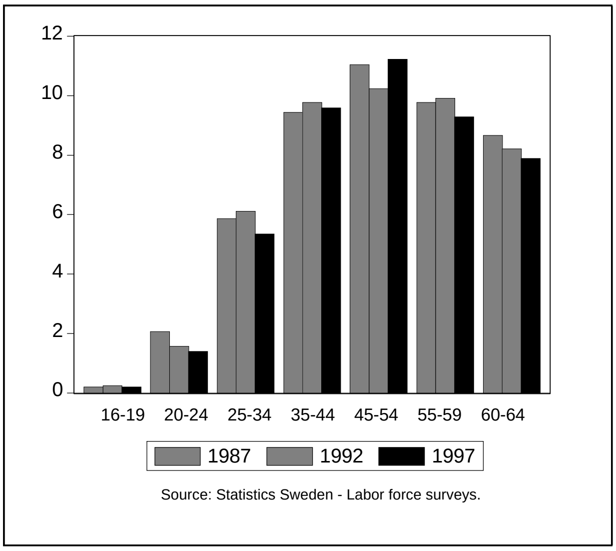
The purpose of this paper is to analyze how decisions to invest in a private project, thereby establishing a new firm, depend on the age and risk preferences of potential entrepreneurs, taking into account individuals' initial level of wealth. We develop an intertemporal portfolio choice model of a risk-averse individual whose

[1] For example, Evans and Jovanovic (1989), Evans and Leighton (1989), and Holtz-Eakin, Joulfaian and Rosen (1994) use US data, Blanchflower and Oswald (1998) use UK data, Lindh and Ohlsson (1996) use Swedish data, and Johansson (1998) uses Finnish data.

[2] Cressy (1996), on the other hand, emphasizes the importance of human capital.

[3] Evans and Leighton (1989) report that, in the US, the "fraction of the labor force that is self-employed increases with age until the early 40s and then remains constant until the retirement years" (p. 520).

Figure 1: Entrepreneurs in percent of total population by age groups, Sweden 1987, 1992 ,1997.



objective is to maximize utility when old (retired). To this end, the individual can at any point in time choose among three assets to invest in: a safe "bank", a mutual fund investing in the stock market, and the private project.

The basic assumption is that potential entrepreneurs face a credit constraint and have to invest a substantial amount of their own capital in the private project, in order to establish a new firm. This is motivated by asymmetric information; in order to be trustworthy and commit themselves to outside investors, they have to commit their own capital.[4] In the model, for simplicity, potential entrepreneurs have to invest *all* of their wealth in the firm in order to receive any gains. The issue is then *either* to invest in the firm *or* to invest in an alternative asset.

A second assumption regarding entrepreneurial activity is that incomplete capital markets, or short sales restrictions, make it impossible to diversify away all risk in the project. The remaining risk is assumed to be described by a Poisson process, such that at random points in time relative large failures occur. Meanwhile, invested capital grows at a fixed rate of return, of which one part could be due to learning-by-doing. Thus, the combination of discrete risk and either-or choice is assumed to

[4] The Swedish Financial Supervisory Authority (*Finansinspektionen* (1995)) analyzed credit ratings of small businesses in Sweden after the financial crisis in the early 1990s. They concluded that there was no lack of risk capital; instead the problems were low solvency ratios and the fact that the firms were small. They also argued that the shift to today's low inflation requires higher solvency and own capital. After the financial crisis, loans against a personal guarantee have become more common.

characterize the problem of investment in small firms.

Of course, alternative investments might comprise any portfolio of traded assets, but we restrict ourselves to two, a safe bank and a risky mutual fund. The market price of the mutual fund is assumed to follow a geometric Brownian motion, that is, a random walk in continuous time.[5]

The investment objects could be interpreted as different occupations available to potential entrepreneurs. Kanbur (1979, 1981) argues that the occupational choice problem should be studied in a discrete choice model, rather than as a traditional portfolio choice problem. In our model, individuals *either* become entrepreneurs, where the return is conditional on their physical and human capital (where at least human capital depreciation is difficult to insure), *or* take one of two possible wage-paying jobs: a safe, low-paid job or a better-paid job. The latter is risky, but without default risk (employees might be better insured).

Moreover, we assume that there are no consumption decisions before the horizon and let instantaneous utility from consumption be zero constant at all points in time. This simplification helps focus on the investment/occupational choice.[6]

Either-or restrictions are examined by Samuelson (1997) in a discrete-time model. The results in our paper confirm his finding that with constant relative risk aversion, the investment decision is independent of the wealth and age (remaining time to a planning horizon) of the investor. Furthermore, we confirm Samuelson's conjecture that, with more realistic utility functions, there is a threshold such that the optimal choice is to invest in the safe asset only if personal wealth is below this threshold. However, whether or not this threshold is increasing or decreasing in age depends on preferences. Gollier and Zeckhauser (1997) show that "convex absolute risk tolerance" is necessary for longer horizons to lead to a higher degree of risk taking. We make use of their results when specifying the utility function.

The results in our paper show that, as the investor gets older, uncertain events are realized which affect accumulation of wealth and thereby the chance of passing any of the thresholds that trigger a change of asset. Note, however, that the

[5] Merton (1971) suggests that models based on Brownian motion are "reasonable models for price behavior of many types of liquid assets" (p. 395), but he uses Poisson processes to model default risk of junk bonds (p. 396). The value of a mutual fund as well as of a private project could be a combination of Brownian motions and Poisson processes. Our assumptions are for simplicity.

[6] This might be a critical assumption if consumption choices are directly dependent on occupation, for example, due to fringe benefits. Granqvist (1998) finds a higher probability of receiving fringe benefits for white-collar labor as compared to blue-collar labor, but the correlation between money wages and fringe benefits is strongly positive. Hence, the rate of return in our analysis could be a sufficient statistic.

thresholds depend on age. For example, if initial wealth is lower than the critical level for investing in the private project, and bank saving is optimal instead, then getting older means that invested capital increases, thereby increasing the chance of becoming an entrepreneur. But this chance is decreased by the fact that the threshold rises as the horizon becomes closer. Hence, age affects the optimal investment strategy in two counteracting ways. In particular, it may be the case that if the risky activity is not undertaken when young, it will never be undertaken.

In our model, the private project can be abandoned before the horizon for two different reasons. Either involuntary exit occurs after a negative shock, interpreted as failure, or the project is successful and the entrepreneur optimally shifts capital to another investment object.

A higher degree of risk aversion implies a higher level of the threshold, at a given age, although it also affects the impact of changes in the age on the threshold. These two effects, in turn, affect the likelihood of investing in risky assets, but possibly in different directions.

The last result reported in this paper is that, if a firm is established, the age of the entrepreneur also affects the expected lifetime of the firm. In the model, the firm is most vulnerable during its first years. After some critical point the survival rate increases.[7] Other things being equal, this critical point is lower for younger entrepreneurs and, hence, the expected lifetime of the firm is higher. The reason for this is a lower opportunity cost for the young; they do not give up their project so easily. This appears to be an unrealistic result. However, *ceteris paribus* means that we compare young and old individuals with the same investment opportunity, with the same rate of return of their project.

The rest of the paper is organized as follows. The model is set up and conditions for an optimal solution are derived in Section 2. Comparative statics with respect to age and risk aversion are introduced in Sections 3 and 4. Section 5 addresses the properties of a firm established under the constraints in the model. The paper ends with concluding remarks. All proofs of the propositions are in the Appendix.

[7] This is consistent with Geroski (1995), who presents stylized facts about entry of new firms into a given market: "The survival rate of most entrants is low, and even successful entrants may take more than a decade to achieve a size comparable to the average incumbent." "... the data suggest that experience may be crucial determinant of survival rates, but that it is not quickly acquired" (p.424).

2 The model

An individual is young (e.g., enters the work force) at time 0 and has an exogenously given initial wealth at that time. Three discrete choices are available at any point in time. The first choice is to invest all current wealth in a "safe bank", yielding a known interest rate, r. The second opportunity is to invest all wealth in a "mutual fund" where the value changes over time, partly by a deterministic trend, partly by continuous disturbances. This is described by a geometric Brownian motion with drift rate α and volatility σ. The third possible choice is to invest all personal wealth in a private project, where the return is described by a Poisson process. If no "bad" events occur, the rate of return is β. If a bad event occurs, wealth is reduced by a fraction θ, $0 < \theta < 1$. It then follows that the expected growth rate is $\beta - \lambda\theta$, and the instantaneous variance is $\lambda\theta^2$, where λdt is the probability of the occurrence of a bad event in the Poisson process during a short interval of time dt.

For simplicity it is assumed, as in Samuelson (1997), that there is no consumption before the horizon and that there is no cost of switching between assets. The problem then is to choose an investment strategy so as to maximize terminal utility at a known date, T. Let $B(W,T)$ denote this utility[8], where W is the realized level of wealth at time T. The optimal investment rules, ω_t^*, $t \in [0,T]$, are found by solving

$$\max_{\{\omega_t\}_0^T} E\left[B(W_T,T)\right]$$

$$\text{s.t.} \quad \frac{dW_t}{W_t} = \begin{cases} rdt & \omega_t = 1 \\ \alpha dt + \sigma dz_t & \omega_t = 2 \\ \beta dt - \theta dq_t & \omega_t = 3 \end{cases} \tag{1}$$

where $r, \alpha, \beta, \sigma, \theta$ are constants, dz_t is the increment of a Wiener process[9] and q_t is a Poisson process such that $dq = 1$ with probability λdt and zero otherwise. ω is a "dummy" variable indicating the choice of asset at time t; $\omega_t = 1$ if bank saving, $\omega_t = 2$ if mutual fund, and $\omega_t = 3$ if all wealth is invested in the private project.

It is assumed that the expected growth rate of the risky assets is higher than the interest rate, i.e.,

$$\alpha > r \quad \text{and} \quad \beta - \theta\lambda > r, \tag{2}$$

[8] The function B can be interpreted as either utility of bequest at the end of life or indirect utility from the period as retired after T.

[9] z_t is a Wiener process if: (i) $z_0 = 0$, (ii) $\{z_t\}$ has continuous trajectories, (iii) $z_t - z_s \sim N\left(0, \sqrt{t-s}\right)$, $s < t$, (iv) $z_t - z_s$ is independent of $z_v - z_u$, $s < t < u < v$ (see Björk (1998, p. 25)).

otherwise risk-averse individuals will always prefer the safe asset.

A specific functional form for the terminal utility is also assumed,

$$B(W,T)=e^{-\rho T}\left(\frac{W^{1-\gamma}}{1-\gamma}+kW\right) \tag{3}$$

where $\gamma>0$ and $k\geq 0$.[10] Below we interpret γ as an approximative measure of risk aversion and, hence, higher γ as higher risk aversion. A positive k complicates the calculations, but later on we show that if $k=0$ then age has no effect on investment strategies, a *myopic* portfolio choice. The assumption of terminal utility is in line with Gollier and Zeckhauser (1997) whose conditions for young investors to take more risk than old investors, other things equal, is satisfied only if $k>0$.

2.1 Solution

The problem is solved by dynamic programming. Since there is no consumption before T, the value function, for a given set of investment rules, is defined as

$$J(W,t)=E_{t,W}\left[B(W_T,T)\right],\qquad \text{all } t\in[0,T]$$

where subscripts denote that expectations are taken conditional on information at time t with realized $W_t=W$. The optimal value function is defined as the expected value at time t of optimal investment rules from t to T,

$$V(W,t)=\sup_{\{\omega_s\}_t^T} J(W,t). \tag{4}$$

By definition, and since $W=0$ at any time implies zero wealth thereafter (see equation (1)), natural boundary conditions are

$$\begin{aligned} V(W,T)&=B(W,T) \qquad && \text{all} \quad W\geq 0 \\ V(0,t)&=B(0,T) && \text{all} \quad t\leq T \end{aligned} \tag{5}$$

The Hamilton-Jacobi-Bellman[11] equation is

$$\sup_{\omega_t}\left\{\frac{1}{dt}E_{t,W}\left[dV(W,t)\right]\right\}=0. \tag{6}$$

The supremum is taken over the portfolio choices at a point in time t when there are only three choices available. V is a stochastic process. Using equation (1) and

[10] A special case is $B(W,T)=e^{-\rho T}(\ln W+kW)$, as $\gamma\rightarrow 1$.

[11] See Øksendal (1998) for a rigorous treatment of H-J-B.

Ito's formula[12] (for $\omega_t = 2$), the expected value of the differential in equation (6) is calculated; under respective choice, substitution yields

$$\sup_{\omega_t} \left\{ V_t^1 + rWV_W^1, \ \ V_t^2 + \alpha W V_W^2 + \tfrac{1}{2}\sigma^2 W^2 V_{WW}^2, \right.$$
$$\left. V_t^3 + \beta W V_W^3 - \lambda \left[V^3(W,t) - V^3((1-\theta)W,t) \right] \right\} = 0, \quad (7)$$

where the notation is $V_t^i = \frac{\partial V(W,t)}{\partial t}$ for $\omega_t = i$, etc. If bank saving is optimal then the first expression in (7) is zero, if the mutual fund is optimal then the second expression is zero and if the private project is an optimal choice then the last expression is zero. That is, the optimal value function is the solution to, at least, one partial differential equation given by (7).

Since the optimal choice is so far unknown, we proceed by first solving all three equations, using the fact that a solution must satisfy the boundary conditions (5). We then have three candidates for the optimal value function:[13]

$$\begin{aligned}
V^1(W,t) &= e^{-\rho T} \left(e^{r(1-\gamma)(T-t)} \frac{W^{1-\gamma}}{1-\gamma} + e^{r(T-t)} kW \right) \\
V^2(W,t) &= e^{-\rho T} \left(e^{\left(\alpha - \frac{1}{2}\sigma^2 \gamma\right)(1-\gamma)(T-t)} \frac{W^{1-\gamma}}{1-\gamma} + e^{\alpha(T-t)} kW \right) \\
V^3(W,t) &= e^{-\rho T} \left(e^{\left(\beta - \lambda\theta - \frac{1}{2}\lambda\theta^2 \eta\gamma\right)(1-\gamma)(T-t)} \frac{W^{1-\gamma}}{1-\gamma} + e^{(\beta-\lambda\theta)(T-t)} kW \right).
\end{aligned}$$

It turns out that the value function is equal to the expected terminal utility of holding the same asset over the whole period, that is, $V^i = E_{t,W}\left[B(W_T,T)\right]$ given $\omega_{\widehat{t}} = i, \forall \widehat{t} \in [t,T]$. This follows from the assumption of no switching costs.

The *optimal* value function is given by

$$V(W,t) = \max\left\{ V^1(W,t), V^2(W,t), V^3(W,t) \right\},$$

to satisfy the definition in equation (4).

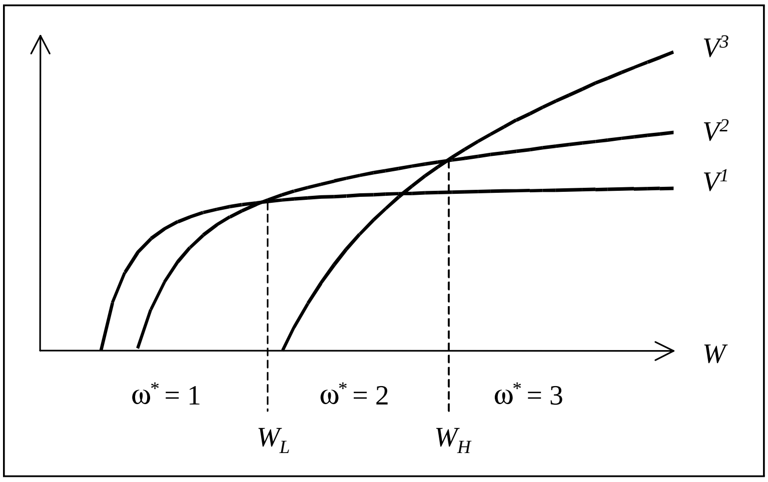
Figure 2 illustrates the solution with a numerical example where each asset is optimal for some W, at a given point in time, t.[14] The optimal choice is implicitly

[12] See, e.g., Björk (1998, p. 38).

[13] Substitution of $e^{-\rho T}\left(a(t)\frac{W^c}{c} + b(t)W\right)$, for arbitrary a, b and c, into (7) gives the solutions. The constant η is introduced for comparability between V^2 and V^3, $\eta \equiv \left(\frac{1-(1-\theta)^{1-\gamma}}{1-\gamma} - \theta\right)\left(\gamma\frac{1}{2}\theta^2\right)^{-1} > 1$, where $\eta \to 1$ as $\theta \to 0$; and $\frac{\partial\eta}{\partial\theta} > 0$, $\frac{\partial\eta}{\partial\gamma} > 0$, all $\theta \in (0,1)$, $\gamma > 0$.

[14] Assumptions are: $\beta = .15, \lambda = .05, \theta = .60, \alpha = .10, \sigma = .2, r = .05, \gamma = 3, k = 1, \rho = .02, T = 100, t = 50, \Rightarrow \{\beta - \lambda\theta = .12, \frac{1}{2}\lambda\theta^2\eta\gamma \approx .10, \frac{1}{2}\sigma^2\gamma = .06\}$.

Figure 2: The optimal value function as the maximum of three graphs



given as depicted on the horizontal axis. $\widehat{W}_L$ and $\widehat{W}_H$ are thresholds that trigger a change in portfolio choice. Note that the optimal choice may not be single valued, for example, at the points of intersection in the figure. The optimal value function is in general neither concave nor differentiable in all points (Samuelson (1997)). It may also be that one or two of the assets are not preferred for any level of W.

2.2 Optimal investment rules

At a given point in time, the real numbers can be divided into, at most, three non-overlapping intervals separated by two threshold values, such that the optimal choice at that time is determined by the interval in which the observed wealth is located. Formally, divide the positive real numbers into three open intervals (*low, medium* and *high*):

$$\mathcal{L}_t = \left(0, \widehat{W}_L(t)\right), \ \mathcal{M}_t = \left(\widehat{W}_L(t), \widehat{W}_H(t)\right), \ \text{and } \mathcal{H}_t = \left(\widehat{W}_H(t), \infty\right),$$

where $0 \leq \widehat{W}_L(t) \leq \widehat{W}_H(t) < \infty$, and refer to wealth as "low", "medium" and "high" for each of the intervals. Note that only $\mathcal{H}_t$ is defined to be necessarily non-empty. The issue now is to derive expressions for the thresholds $\widehat{W}_L(t)$ and $\widehat{W}_H(t)$ so that the optimal rule is the same within each interval.[15]

The special case where $k = 0$ in the terminal utility function (3) gives the following result.

[15] These expressions are derived and defined in the Appendix, which also contains the proofs of the lemmas and propositions.

Proposition 1 *If preferences are characterized by constant relative risk aversion* ($k = 0$)*, then the optimal investment choice is independent of time and the level of wealth.*

In this case, an investor's strategy is totally independent of the time left to the planning horizon, which might be unrealistic. In the remainder of the paper, we therefore assume that $k > 0$. This complicates the calculations, but makes it possible to analyze the dynamic effects on investments.[16]

Lemma 1 $\mathcal{L}_t$ *is non-empty if and only if*

$$r > \max\left\{\alpha - \tfrac{1}{2}\sigma^2\gamma,\ \beta - \lambda\theta - \tfrac{1}{2}\lambda\theta^2\eta\gamma\right\}, \tag{8}$$

which is equivalent to

$$\gamma > \max\left\{\frac{\alpha - r}{\frac{1}{2}\sigma^2}, \frac{\beta - \lambda\theta - r}{\frac{1}{2}\lambda\theta^2\eta}\right\}. \tag{9}$$

Proposition 2 *A safe asset is optimal if and only if wealth is low, i.e.,* $W \in \mathcal{L}_t$.

Thus, in a population of individuals who are identical except for wealth, we suggest that if there is positive demand for the riskfree asset, this demand emanates from those with the lowest levels of wealth. A further prerequisite is that these individuals are sufficiently risk averse.[17]

Proposition 3 *If income is high,* $W \in \mathcal{H}_t$*, then it is optimal to invest in the project with the highest expected growth rate, i.e., in the mutual fund if* $\alpha > \beta - \lambda\theta$ *and in the private project if* $\alpha < \beta - \lambda\theta$.

Individuals with high wealth prefer the asset with the highest expected growth rate. This is a consequence of the assumed preferences. For high levels of wealth, the individual is risk neutral in the Arrow-Pratt sense and only the expected return matters. By assumption (2), the choice, then, is between the two risky assets. If the expected growth rates are equal, then the private project is undertaken only if it has lower volatility.[18]

[16] If $k = 0$, then the stock market is preferred to the safe asset iff risk aversion is low enough, i.e., $\gamma \leq 2(\alpha - r)/\sigma^2$; cf. Samuelson (1997, p. 190). $k > 0$ implies convex risk tolerance ($-B'/B''$), a necessary condition for young people to take more risk than older people according to Gollier and Zeckhauser (1997).

[17] Risk neutrality, $\gamma = 0$, violates equation (8) by the assumption in equation (2).

[18] If $\alpha = \lambda\theta$, then the private project is established if $\lambda\theta^2\eta < \sigma^2 \Rightarrow \lambda\theta^2 < \sigma^2$ (lower variance), and an individual is indifferent if $\lambda\theta^2\eta = \sigma^2$.

For example, in Figure 2 above, the private project is assumed to have the highest expected growth rate and is preferred by those with relatively high wealth; saving in a bank is preferred if wealth is low. Individuals with medium wealth prefer the mutual fund, under the specific parameter assumptions.

However, it might also be the case that one of the risky assets is not optimal for any level of wealth, that is, $\mathcal{M}_t$ is empty.

Lemma 2 *$\mathcal{M}_t$ is non-empty if and only if*

$$\begin{aligned} 0 &< (\beta - \lambda\theta) - \alpha < \tfrac{1}{2}\left(\lambda\theta^2\eta - \sigma^2\right)\gamma, \quad \textit{or} \\ 0 &< \alpha - (\beta - \lambda\theta) < \tfrac{1}{2}\left(\sigma^2 - \lambda\theta^2\eta\right)\gamma. \end{aligned}$$

It is of particular interest to examine under what conditions the private project will be undertaken. It is evident from Proposition 3 that investing in the private project is optimal for some levels of wealth if the expected rate of return is higher than the expected rate of return on the alternative investment in the stock market. If this is not the case then there is a positive probability of undertaking the private project only if the difference in expected rates of return is not too large.

Proposition 4 *If the expected growth rate of the private project is low, $\beta - \lambda\theta < \alpha$, then the private project is undertaken if and only if $W \in \mathcal{M}_t$, under the condition that the difference in expected growth rates between the private project and the mutual fund is sufficiently small,*

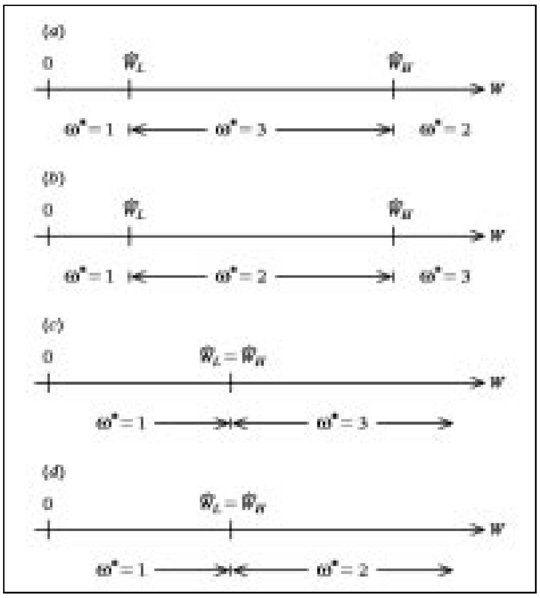
$$\alpha - (\beta - \lambda\theta) < \tfrac{1}{2}\left(\sigma^2 - \lambda\theta^2\eta\right)\gamma, \tag{10}$$

which requires the project to have lower volatility, $\lambda\theta^2 < \sigma^2$.

Figure 3 summarizes all possible cases if the lower threshold is strictly positive. In case b, for example, with low wealth, capital is saved in a bank, due to risk aversion. For a medium level of wealth it is optimal to take the risk in the stock market. If the individual's loss in the stock market is too large, it is possible and optimal to switch back to bank savings. Starting a private project requires a high level of wealth. The project has relatively high risk (variance) but also a high expected rate of return in this case.

In case a the private project is undertaken for an intermediate level of wealth which implies that there are two ways to exit: a "low exit" at $\widehat{W}_L$ where the individual switches to the safe asset if an established project fails, and a "high exit" at $\widehat{W}_H$

Figure 3: Possible solutions at a given point in time assuming $\widehat{W}_L > 0$



where a switch to the mutual fund occurs. In the occupational choice interpretation, the entrepreneur liquidates the private firm at free will and takes a job. Hence, there can be one exit due to failure and another due to success.

Cases c and d occur when one of the risky assets "dominates" the other (Lemma 2). In the case of no demand for the safe asset (Proposition 2), there are four cases similar to those in Figure 3, the only difference is that $\widehat{W}_L = 0$. The fact that the conditions in the propositions depend only on the parameters gives us the following corollary.

Corollary 1 *If, at any point in time, there is a positive level of wealth such that a specific asset is preferred, then it is preferred at some level of wealth at all points in time. Moreover, the order in which the assets are preferred is independent of time.*

Hence, the "ranking" of the assets, that is, one of the cases in Figure 3, is the same over the individual's lifetime. This implies that the thresholds cannot intersect in the time-wealth space. However, they are not constant over time. In the next section we discuss dependence of the age of an investor, that is, the time left to the horizon $(T - t)$.

3 Age dynamics

There are two dynamic effects of age on investments in private projects and thereby the establishment of small firms. First, there are opportunities to accumulate capital over time. Second, risk aversion and thereby the thresholds might change as the planning horizon gets closer. In particular, it might be the case that if the risky activity is not undertaken when young, it will never be taken.

The thresholds defined above are continuous functions of time. We now derive conditions to determine whether or not the thresholds are increasing or decreasing over time, and the magnitude of the changes. An increasing lower threshold, for example, means that the threshold is low for younger individuals, because there is time left to reallocate the portfolio if too many bad events occur. As time passes, wealth is increasing at a rate of return that depends on the asset chosen and realizations of random events. Accordingly, the magnitude of an increase or decrease in the thresholds will affect the likelihood that a change in the investment will actually occur.

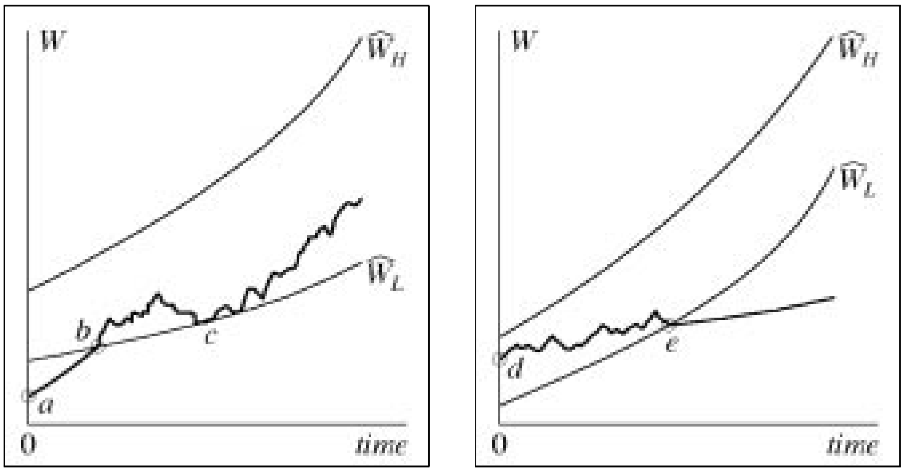
It follows from the preceding section that with low initial wealth, risk aversion implies that money is initially saved in a bank. Then invested capital grows over time, which increases the likelihood of switching to a risky investment before the horizon. But the problem here is that the thresholds might also increase as the individual gets older.

Proposition 5 *An individual with low initial wealth, below $\widehat{W}_L(0)$, will switch from the safe asset to a risky asset only if the relative increase in the lowest threshold is less than the safe interest rate; this condition is fulfilled if and only if risk aversion (γ) is sufficiently high.*

The Proposition is explained in Figure 4, where it is assumed that the low threshold triggers a change to the mutual fund. When initial wealth is low (e.g., point a in 4(a)), wealth follows a deterministic path which is increasing exponentially with the rate of return of the safe asset, r. To be able to reach the lowest threshold, it cannot grow at a higher rate than r.

Suppose this is the case, and the lower threshold is eventually reached (point b in 4(a)); it then triggers a change from bank saving to a mutual fund. Since there are no switching costs, if the threshold is reached from above (at c in 4(a)) then the individual changes to bank saving – but then instantaneously reaches the threshold

Figure 4: Examples of time paths.



(a) "Reflecting" threshold. **(b) "Absorbing" threshold.**

from below, and thus immediately changes back to the mutual fund.[19]

The condition in the proposition is independent of time. Hence if it holds at one age, it will always hold. This implies that the probability of leaving the mutual fund for bank saving before the horizon is zero. In technical terms, the threshold is a lower "reflecting" barrier to the stochastic process of the mutual fund (cf. Dixit (1993)). It can be thought of as endogenously given irreversibility of the investment, with capital trapped in one investment.

On the other hand, if the threshold grows faster than the interest rate and initial income is low then, of course, the probability of changing investment is zero. In the occupational choice interpretation, an individual is trapped in the safe occupation. Furthermore, even if initial wealth is at a medium level (e.g., point d in Figure 4(b)), there is a chance of reaching the low threshold, depending on random events. If this occurs before the horizon (point e in 4(b)), then it triggers a change to the safe asset and the prediction is that the individual never switches from the safe asset to a risky one. That is, the threshold is an "absorbing" barrier to the stochastic process.

A necessary condition for the private project to survive over a long period of time is that the entrepreneur does not give up the project for some alternative investment. In terms of the model, the threshold that triggers a shift into entrepreneurship is not increasing faster than the rate of return on the private project when no shocks

[19] Same policy can be optimal in certain real investment problems, a "barrier control policy", see Dixit and Pindyck (1994, p.362).

occur.

Proposition 6 *If the private project is undertaken by an individual with medium wealth (cases a and c) and if the relative increase in the lower threshold is less than the safe interest rate, then "low exit" to the safe asset only occurs if the private project fails, i.e., a bad event is realized.*

This implies that individuals with low initial wealth (and a sufficiently long horizon) are able to build up their capital stock in the safe asset and then eventually shift to the private project, where they stay until either a bad event occurs or they liquidate the firm, which could happen either at the horizon or when the alternative risky asset becomes attractive.

Proposition 7 *If the private project is undertaken by an individual with high wealth (case b) and sufficiently high risk aversion (γ), then he does not exit unless the project fails.*

4 Risk aversion

We may now derive the characteristics of risk aversion before the terminal date. We also show the consequence of the relatively complicated utility function which is assumed, instead of the common constant relative risk aversion function. Moreover, the effects of risk aversion on the minimum age (of individuals with low initial wealth) to invest in risky assets are derived.

The Arrow-Pratt measure of relative risk aversion at time t, is defined as

$$R_R^i(W,t) = -\frac{V_{WW}^i(W,t)W}{V_W^i(W,t)} \quad \text{for } \omega_t = i,$$

which is decreasing in W if and only if $k > 0$, and where $R_R^i(W,t) \to \frac{1}{1+kW^\gamma}\gamma \leq \gamma$, as $(T-t) \to 0$.

Proposition 8 $R_R^i(W,t) \to 0$, *for all* $W > 0$ *and all choices* ω_t, *as* $(T-t) \to \infty$ *if and only if risk aversion* (γ) *is sufficiently low.*

In this case, risk aversion is lower for younger persons than for those close to the horizon. This is consistent with the findings of Gollier and Zeckhauser (1997). With an infinite horizon, individuals behave as if they were risk neutral. The reason is that if realization of the risky assets turns out to be bad, there is infinite time for reallocations of the portfolio. Close to the horizon, preferences as retired (or utility generated by bequest) dominate the reallocation options.

It was shown in Proposition 5 that if initial wealth is relatively low, then contrary to Proposition 8, it is necessary to have *high* enough risk aversion to replace bank saving with a risky asset. This may be counterintuitive but has a natural explanation. Increasing risk aversion not only decreases the growth rate, but also increases the *level* of the threshold at a given age. One way to measure the net effect is to derive the time it takes for an individual to save enough money in the bank to be willing to take risks. This time, denoted t_1, is deterministic and is implicitly given by

$$W_0 e^{rt_1} = \widehat{W}_L(t_1), \qquad \text{if } t_1 < T \tag{11}$$

where initial wealth W_0 is assumed to be less than $\widehat{W}_L(0)$.

Proposition 9 *If t_1 defined in (11) exists, then t_1 is smaller if γ is lower, that is, a less risk-averse individual will invest in a risky asset for the first time at a younger age.*

5 Properties of the firm

If a firm is established when the private project is undertaken, then it is of interest to examine how the age of the *investor* affects the properties of the firm. One way of measuring the success of a firm is to examine its expected lifetime.

Note that a firm, created at a particular point in time $\widehat{t}$, survives a single shock only if the age of the *firm* is at least $t^* - \widehat{t}$, where t^* is the smallest $t \geq \widehat{t}$, satisfying

$$(1-\theta) \cdot W(t) \geq \widehat{W}(t). \tag{12}$$

Moreover, the instantaneous hazard rate of firm exit is constant up to time t^* and given by the probability of a Poisson event (λdt). After that time, more than one shock is required for the firm to be abandoned. For this reason the hazard rate decreases. Thus, the closer t^* is to $\widehat{t}$, the larger is the chance that the firm will survive and the longer the expected lifetime of the firm.

Surviving to time t^* is equivalent to no shocks occurring in the interval $\left[\widehat{t}, t^*\right]$ and, if no shocks occur, then wealth invested in a firm increases at the rate β.

Thus, creating the firm at time $\widehat{t}$, investing $W(\widehat{t}) = \widehat{W}(\widehat{t})$ of own funds, implies that $W(t^*) = e^{\beta\left(t^*-\widehat{t}\right)}\widehat{W}(\widehat{t})$. Substitution into (12) yields t^* implicitly given by

$$\ln\left(1-\theta\right) + \beta\left(t^* - \widehat{t}\right) - \left(\ln\widehat{W}(t^*) - \ln\widehat{W}(\widehat{t})\right) = 0. \tag{13}$$

Interpreting $\widehat{t}$ as the age of the investor at the time of the firm's establishment gives the following proposition.

Proposition 10 *A firm established by a relatively young entrepreneur has a longer expected lifetime compared to a firm established by an older entrepreneur, ceteris paribus, only if risk aversion (γ) is sufficiently high (such that the condition for exit only in case of failure is fulfilled).*

The reason is that if the condition holds then the growth rate of the thresholds increases with age (the logarithmic value of the threshold is a convex function of time) whereas wealth grows at a constant rate. As a result, a younger individual can more quickly build up a sufficient buffer to survive a negative shock. The buffer is both smaller (since the threshold is lower) and "easier" to build up, since the threshold increases more slowly with longer time to the horizon.

6 Concluding remarks

The model developed here should be regarded as partial analysis of a single individual in a world populated by heterogeneous individuals. Extensions of this framework could include different private projects distributed, randomly or not, in some way, as well as heterogeneous agents who are given initial wealth and risk preferences in some specific way.

Of course, the restrictions in the paper could be relaxed, for example by allowing linear combinations of alternative assets. Preliminary results, however, suggest that in our case, when a private project is undertaken by an individual with high wealth, the optimal investment results of the private project still hold under a short sales restriction. Individual agents could be assumed to take market conditions as given, that is, the rate of return on the riskfree asset and rate of return and volatility of the alternative risky asset. Then these parameters could, in principle, be determined by market equilibrium. But the risk of private projects could still be assumed to be undiversified.

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Appendix

A Derivation of the thresholds

The thresholds required for the optimal rule to be the same within each interval are derived as follows. Write the value functions as

$$V^i(W,t) = e^{-\rho T}\left(h_i(t)\frac{W^{1-\gamma}}{1-\gamma} + g_i(t)\,W\right), \qquad \omega_t = i,$$

$i = 1, 2, 3,$ where

$$\begin{aligned} h_1(t) &= e^{r(1-\gamma)(T-t)}, & g_1(t) &= ke^{r(T-t)}, \\ h_2(t) &= e^{\left(\alpha-\frac{1}{2}\sigma^2\gamma\right)(1-\gamma)(T-t)}, & g_2(t) &= ke^{\alpha(T-t)}, \\ h_3(t) &= e^{\left(\beta-\lambda\theta-\frac{1}{2}\lambda\theta^2\eta\gamma\right)(1-\gamma)(T-t)}, & g_3(t) &= ke^{(\beta-\lambda\theta)(T-t)}. \end{aligned} \tag{A1}$$

To find the optimal choice, fix a time t and define the (possibly empty) set $\{W_{i,j}\}$ by those positive W which imply that asset i is preferred to j. Any $W_{i,j}$ satisfies $V^i(W_{i,j},t) \geq V^j(W_{i,j},t)$, that is,

$$h_i(t)\frac{W_{i,j}^{1-\gamma}}{1-\gamma} + g_i(t)\,W_{i,j} \geq h_j(t)\frac{W_{i,j}^{1-\gamma}}{1-\gamma} + g_j(t)\,W_{i,j} \quad i,j = 1,2,3. \tag{A2}$$

If $\frac{h_i(t)}{1-\gamma} > \frac{h_j(t)}{1-\gamma}$ then solving for $W_{i,j}$ in equation $(A2)$ yields

$$\begin{array}{ll} 0 < W_{i,j} < \infty & \text{if } \; g_i(t) \geq g_j(t) \\ 0 < W_{i,j} \leq \left(\frac{h_i(t)-h_j(t)}{(1-\gamma)(g_j(t)-g_i(t))}\right)^{\frac{1}{\gamma}} & \text{if } \; g_i(t) < g_j(t)\,. \end{array} \tag{A3}$$

That is, binary relations can be derived between any two portfolio choices. The optimal investment rule is obtained by implementing the transitivity axiom of preferences. The lower and higher thresholds may then be defined as follows.

Definition A1 *The lower threshold is defined as* $(k>0)$

$$\widehat{W}_L(t) = \begin{cases} e^{-r(T-t)}\left(\dfrac{\frac{1}{1-\gamma}\left(1-e^{-\left(r-\alpha+\frac{1}{2}\sigma^2\gamma\right)(1-\gamma)(T-t)}\right)}{k\left(e^{(\alpha-r)(T-t)}-1\right)}\right)^{\frac{1}{\gamma}} \\ \qquad\qquad \textit{if } \; r > \alpha-\frac{1}{2}\sigma^2\gamma > \beta-\lambda\theta-\frac{1}{2}\lambda\theta^2\eta\gamma, \\ e^{-r(T-t)}\left(\dfrac{\frac{1}{1-\gamma}\left(1-e^{-\left(r-(\beta-\lambda\theta)+\frac{1}{2}\lambda\theta^2\eta\gamma\right)(1-\gamma)(T-t)}\right)}{k\left(e^{(\beta-\lambda\theta-r)(T-t)}-1\right)}\right)^{\frac{1}{\gamma}} \\ \qquad\qquad \textit{if } \; r > \beta-\lambda\theta-\frac{1}{2}\lambda\theta^2\eta\gamma > \alpha-\frac{1}{2}\sigma^2\gamma, \\ 0 \quad \textit{otherwise.} \end{cases}$$

Definition A2 *The higher threshold is defined as* $(k > 0)$

$$\widehat{W}_H(t) = \begin{cases} \left(\dfrac{e^{\left(\alpha - \frac{1}{2}\sigma^2\gamma\right)(1-\gamma)(T-t)} - e^{\left(\beta - \lambda\theta - \frac{1}{2}\lambda\theta^2\eta\gamma\right)(1-\gamma)(T-t)}}{(1-\gamma)\,k\left(e^{(\beta-\lambda\theta)(T-t)} - e^{\alpha(T-t)}\right)} \right)^{\frac{1}{\gamma}} & \\ \qquad \text{if } 0 < (\beta - \lambda\theta) - \alpha < \frac{1}{2}\left(\lambda\theta^2\eta - \sigma^2\right)\gamma & \\ \qquad \text{or } 0 < \alpha - (\beta - \lambda\theta) < \frac{1}{2}\left(\sigma^2 - \lambda\theta^2\eta\right)\gamma, & \\ \widehat{W}_L(t) \quad \text{otherwise.} & \end{cases}$$

The definitions are consistent with $0 \le \widehat{W}_L(t) \le \widehat{W}_H(t) < \infty$.

B Proof of Proposition 1

If $k = 0$, then the value function takes the form

$$V^i(W,t) = e^{-\rho T} h_i(t) \frac{W^{1-\gamma}}{1-\gamma} \qquad i = 1,2,3,$$

where $e^{-\rho T} W^{1-\gamma} > 0$. The optimal investment choice is then determined by $\max_{\omega}\left\{\frac{h_i(t)}{1-\gamma}\right\}$, which is equivalent to

$$\max_{\omega}\left\{ r,\ \alpha - \tfrac{1}{2}\sigma^2\gamma,\ \beta - \lambda\theta - \frac{1}{2}\lambda\theta^2\eta\gamma \right\}.$$

Hence it is independent of time and wealth.

C Sketched proof of Lemma 1 and Proposition 2

Note that condition (8) is equivalent to

$$\frac{h_1(t)}{1-\gamma} > \frac{h_j(t)}{1-\gamma}, \quad j = 2,3 \text{ all } t < T \tag{A4}$$

and the assumption in equation (2) is equivalent to

$$g_1(t) < g_j(t), \quad j = 2,3 \text{ all } t < T. \tag{A5}$$

From (A3) (suppressing the time index): asset 1 is preferred for $W \in \left(0, \widehat{W}_L\right)$ where

$$\widehat{W}_L = \min_{j=2,3}\left(\frac{h_1 - h_j}{(1-\gamma)(g_j - g_1)}\right)^{\frac{1}{\gamma}}. \tag{A6}$$

"Min" is due to the transitivity axiom of preferences. Suppose that $\frac{h_1-h_2}{(1-\gamma)(g_2-g_1)} < \frac{h_1-h_3}{(1-\gamma)(g_3-g_1)}$. Then asset 1 is preferred to *both* assets 2 and 3 (denoted $1 \succ 3$, $1 \succ 2$) iff $W < \widehat{W}_L$, and if $W \in \left(\widehat{W}_L, \frac{h_1-h_3}{(1-\gamma)(g_3-g_1)}\right)$ then $1 \succ 3$, $2 \succ 1$. However, by transitivity, $2 \succ 1 \succ 3$ ($W > \frac{h_1-h_3}{(1-\gamma)(g_3-g_1)} \Rightarrow 3 \succ 1$, $2 \succ 1$), i.e., $\widehat{W}_L < \infty$. $\widehat{W}_L > 0$ from (A4) and (A5). Substitution from (A1) into (A6) yields the equations in Definition A1.

D Proof of Proposition 3

It is sufficient to show that the choice for a high level of wealth is determined by the relation between g_2 and g_3 as defined in equation (A1). Proposition 2 implies that the safe asset is optimal only in an interval bounded from above. The proof follows from equation (A3): $\left\{g_i > g_j \text{ and } \frac{h_i(t)}{1-\gamma} > \frac{h_j(t)}{1-\gamma}\right\} \Rightarrow i \succ j$ for all W; $\left\{g_i > g_j \text{ and } \frac{h_i(t)}{1-\gamma} \leq \frac{h_j(t)}{1-\gamma}\right\} \Rightarrow i \succ j$ if $W > \left(\frac{h_j - h_i}{(1-\gamma)(g_i - g_j)}\right)^{\frac{1}{\gamma}} < \infty$.

E Sketched proof of Lemma 2 and Proposition 4

$\mathcal{M}_t$ is non-empty if there exists a positive $\widehat{W}_H < \infty$ such that only if $W = \widehat{W}_H$ are individuals indifferent between assets 2 and 3. Note that

$$\begin{aligned} (\beta - \lambda\theta) - \alpha < \tfrac{1}{2}\left(\lambda\theta^2\eta - \sigma^2\right)\gamma \quad &\Leftrightarrow \quad \frac{h_3(t)}{1-\gamma} < \frac{h_2(t)}{1-\gamma}, \\ 0 < (\beta - \lambda\theta) - \alpha \quad &\Leftrightarrow \quad g_3(t) > g_2(t), \quad \forall t < T. \end{aligned}$$

Equation (A3) then implies that such $\widehat{W}_H = \left(\frac{h_2 - h_3}{(1-\gamma)(g_3 - g_2)}\right)^{1/\gamma} < \infty$, where $\widehat{W}_H > \widehat{W}_L$ from Propositions 2 and 3. The opposite case would also have to be checked, i.e., $0 < \alpha - (\beta - \lambda\theta) < \frac{1}{2}\left(\sigma^2 - \lambda\theta^2\eta\right)\gamma$, and the predictions from equation (A3).

F Proof of Propositions 5 – 7

$W < \widehat{W}_L \Rightarrow \frac{\partial \ln W}{\partial t} = r$ with probability one; hence, the threshold has to grow at a lower rate in order to be reached. Logarithmic differentiation w.r.t. t of $\widehat{W}_L(t) > 0$ and $\widehat{W}_H(t) > 0$, given in Definitions A1 and A2, yields,

$$\frac{\partial \ln \widehat{W}_L(t)}{\partial t} = r - \frac{1}{\gamma}\left(\frac{(1-\gamma)B}{e^{(1-\gamma)B(T-t)} - 1} - \frac{A}{1 - e^{-A(T-t)}}\right), \tag{A7}$$

$$\frac{\partial \ln \widehat{W}_H(t)}{\partial t} = M - \frac{1}{\gamma}\left(\frac{(1-\gamma)D}{e^{(1-\gamma)D(T-t)} - 1} - \frac{C}{1 - e^{-C(T-t)}}\right), \tag{A8}$$

all $t \in [0, T)$, where

$$
\begin{aligned}
A &\equiv \begin{cases} \alpha - r, & \text{cases: } b, d \\ \beta - \lambda\theta - r, & \text{cases: } a, c \end{cases} \\
B &\equiv \begin{cases} \frac{1}{2}\sigma^2\gamma - A, & \text{cases: } b, d \\ \frac{1}{2}\lambda\theta^2\eta\gamma - A, & \text{cases: } a, c \end{cases} \\
C &\equiv |\beta - \lambda\theta - \alpha| \\
D &\equiv \left|\tfrac{1}{2}\left(\lambda\theta^2\eta - \sigma^2\right)\gamma - C\right| \\
M &\equiv \begin{cases} \alpha + \frac{1}{2}\sigma^2(1-\gamma), & \text{case: } b \\ \beta - \lambda\theta + \frac{1}{2}\lambda\theta^2\eta(1-\gamma), & \text{case: } a \end{cases}
\end{aligned}
$$

and where "cases" refer to Figure 3.

Proof of Proposition 5: We show that whether $\frac{\partial \ln \widehat{W}_L(t)}{\partial t} \leq r$ depends only on the parameters (in particular γ). From (A7) i follows that,

$$
\begin{aligned}
\frac{\partial \ln \widehat{W}_L(t)}{\partial t} \leq r \Leftrightarrow \frac{(1-\gamma)B}{e^{(1-\gamma)B(T-t)} - 1} &\geq \frac{A}{1 - e^{-A(T-t)}} \Leftrightarrow \\
\Leftrightarrow \frac{-(1-\gamma)B(T-t)}{1 - e^{-(-(1-\gamma)B)(T-t)}} &\geq \frac{A(T-t)}{1 - e^{-A(T-t)}} \Leftrightarrow -(1-\gamma)B \geq A,
\end{aligned}
$$

where the last step follows from the fact that $\frac{\partial}{\partial x}\left(\frac{x}{1-e^{-x}}\right) > 0$, $\forall x \neq 0$, where $A, B > 0$ by assumption. Written out, this is fulfilled if and only if

$$
\gamma \geq \begin{cases} 1 + \dfrac{\alpha - r}{\frac{1}{2}\sigma^2} & \text{cases: } b, d, \\ 1 + \dfrac{\beta - \lambda\theta - r}{\frac{1}{2}\lambda\theta^2\eta} & \text{cases: } a, c. \end{cases} \tag{A9}
$$

Proof of Proposition 6: $\frac{\partial \ln \widehat{W}_L}{\partial t} < r \Rightarrow \frac{\partial \ln \widehat{W}_L}{\partial t} < \beta$ since $r < \beta - \lambda\theta$ by assumption and $\beta - \lambda\theta < \beta$. Once a project is established, invested capital will grow faster than the threshold, unless the project fails.

Proof of Proposition 7: Following the same steps as above, it can be shown that a necessary and sufficient condition for $\frac{\partial \ln \widehat{W}_H}{\partial t} \leq \alpha + \frac{1}{2}\sigma^2(1-\gamma)$ is

$$
\gamma > 1 + \frac{\beta - \lambda\theta - \alpha}{\frac{1}{2}\left(\lambda\theta^2\eta - \sigma^2\right)}, \tag{A10}
$$

and, in case b, $\frac{\beta - \lambda\theta - \alpha}{\frac{1}{2}\left(\lambda\theta^2\eta - \sigma^2\right)} > 0$. Hence $\gamma > 1$ which implies that $\frac{\partial \ln \widehat{W}_H}{\partial t} < \beta$ since

$$
\alpha + \tfrac{1}{2}\sigma^2(1-\gamma) < \alpha < \beta - \lambda\theta < \beta.
$$

G Proof of Proposition 8

$$R_R(W,t) = \frac{1}{1+\frac{g(t)}{h(t)}W^{\gamma}}\gamma$$

$$\frac{g_1(t)}{h_1(t)} = k \cdot e^{r\gamma(T-t)} \to \infty \quad \text{when } (T-t) \to \infty$$

$$\frac{g_2(t)}{h_2(t)} = k \cdot e^{\left(\alpha+\frac{1}{2}\sigma^2(1-\gamma)\right)\gamma(T-t)}$$

$$\begin{cases} \to 0 & \text{iff } \; -(1-\gamma) > \frac{\alpha}{\frac{1}{2}\sigma^2} \\ \to \infty & \text{iff } \; -(1-\gamma) < \frac{\alpha}{\frac{1}{2}\sigma^2} \end{cases} \quad \text{when } \; (T-t) \to \infty$$

$$\frac{g_3(t)}{h_3(t)} = k \cdot e^{\left(\beta-\lambda\theta+\frac{1}{2}\lambda\theta^2\eta(1-\gamma)\right)\gamma(T-t)}$$

$$\begin{cases} \to 0 & \text{iff } \; -(1-\gamma) > \frac{\beta-\lambda\theta}{\frac{1}{2}\lambda\theta^2\eta} \\ \to \infty & \text{iff } \; -(1-\gamma) < \frac{\beta-\lambda\theta}{\frac{1}{2}\lambda\theta^2\eta} \end{cases} \quad \text{when } \; (T-t) \to \infty$$

hence $\gamma < \min\left\{1+\frac{\alpha}{\frac{1}{2}\sigma^2},\ 1+\frac{\beta-\lambda\theta}{\frac{1}{2}\lambda\theta^2\eta}\right\}$ for $R_R \to 0$ when $(T-t) \to 0$ in all cases.

H Proof of Proposition 9

Implicit differentiation of equation (11) yields

$$\frac{dt_1}{d\gamma} = -\frac{\frac{\partial \widehat{W}_L(t_1)}{\partial \gamma}}{\frac{\partial \widehat{W}_L(t_1)}{\partial t} - r\widehat{W}_L(t_1)} > 0.$$

The inequality follows from the facts that the numerator is positive and $\frac{\partial \widehat{W}_L(t_1)}{\partial t} - r\widehat{W}_L(t_1) < 0 \Leftrightarrow \frac{\partial \ln \widehat{W}_L(t_1)}{\partial t} < r$, which follows from Proposition 5, since otherwise $t_1 < T$ does not exist.

I Proof of Proposition 10

Define $w(t) \equiv \ln W(t)$. The proof then follows from the following lemma.

Lemma A1 *$t^* - \widehat{t}$ is increasing in $\widehat{t}$ iff $\widehat{w}$ is a convex function.*

Proof Implicit differentiation of (13) yields $\frac{d\left(t^*-\widehat{t}\right)}{d\widehat{t}} = \frac{\widehat{w}'(t^*)-\widehat{w}'(\widehat{t})}{\beta-\widehat{w}'(t^*)}$, where the numerator is positive iff $\widehat{w}$ is convex and $\beta > \widehat{w}'(t^*)$ if the project is undertaken. $\square$

It remains to prove that condition (A9) and (A10) are necessary and sufficient for $w(t)$ to be a convex function. Differentiation of (A7) yields

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln \widehat{W}_L(t)}{\partial t}\right) =$$

$$= -\frac{1}{\gamma(T-t)^2}\left(\frac{(-(1-\gamma)B(T-t))^2 e^{(1-\gamma)B(T-t)}}{\left(1-e^{(1-\gamma)B(T-t)}\right)^2} - \frac{(A(T-t))^2 e^{-A(T-t)}}{\left(1-e^{-A(T-t)}\right)^2}\right),$$

and by considering the function g:

$$g(x) = \frac{x^2 e^{-x}}{(1-e^{-x})^2} \quad \Rightarrow \quad g'(x) < 0, \;\; x > 0.$$

it is proven that

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln \widehat{W}_L(t)}{\partial t}\right) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{(-(1-\gamma)B(T-t))^2 e^{(1-\gamma)B(T-t)}}{\left(1-e^{(1-\gamma)B(T-t)}\right)^2} \leq \frac{(A(T-t))^2 e^{-A(T-t)}}{\left(1-e^{-A(T-t)}\right)^2}$$

$$\Leftrightarrow \quad -(1-\gamma)B \geq A \Leftrightarrow \text{Condition (A9)},$$

since $A(T-t) > 0$. (Similar exercise with (A8) yields $\frac{\partial}{\partial t}\left(\frac{\partial \ln \widehat{W}_L(t)}{\partial t}\right) \geq 0$ iff condition (A10) holds).